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Topological representations of quantum groups and conformal field theory

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These are three introductory lectures on the relation between representations of affine Kac–Moody algebras, homology of configuration spaces with local coefficient systems, and quantum groups. The first lecture contains background on highest weight representations of affine Kac–Moody algebras. In the second lecture, conformal blocks, the Friedan–Shenker connection and the Knizhnik–Zamolodchikov (KZ) equation are reviewed. In the third lecture, the case of sl_2 is studied in more detail. Integral representations of solutions of the KZ equation are derived, and recent results, obtained in collaboration with C. Wieczerkowski, on the relation between integration cycles and representations of $U_q(sl_2)$ are explained.

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1. Highest weight representations of affine untwisted Kac–Moody Lie algebras

We start from the following data:

(i) A simple complex Lie algebra g with invariant bilinear form $(\ ,\ )$ normalized in such a way that the highest root has norm squared 2, and with dual Coxeter number $h^\vee$. For sl_N, $(X,Y) = -\mathrm{tr}(XY)$, and $h^\vee = N$.

(ii) A complex number $k \neq h^\vee$ called level.

1.1. LOOP ALGEBRAS AND THEIR CENTRAL EXTENSION

The loop algebra Lg is defined to be the Lie algebra $g \otimes \mathbb{C}((t))$ of formal Laurent series in the parameter t with coefficients in g and only finitely many negative powers of t. We have the decomposition $g = n_- \oplus g \oplus n_+$, where $n_+ = g \otimes t\mathbb{C}[[t]]$ and $n_- = g \otimes t^{-1}\mathbb{C}[t^{-1}]$.

The central extension $L\hat{g}_k$ is the Lie algebra $Lg \oplus \mathbb{C}$ with Lie brackets

$$[X \otimes f(t) \oplus \zeta, Y \otimes g(t) \oplus \eta] = [X,Y] \otimes f(t)g(t) \oplus (X,Y)k\,\mathrm{res}(f'(t)g(t)\mathrm{d}t)\,.$$

The residue is defined as $\mathrm{res}(\sum a_n t^n \mathrm{d}t) = a_{-1}$. Taking the derivative with respect to t defines a derivation of $L\hat{g}_k$. We can thus adjoin an element L_{-1} to the Lie algebra such that

$$[L_{-1}, X \otimes f(t) \oplus \zeta] = -X \otimes f'(t).$$

The polynomial subalgebra $L^{\mathrm{pol}}\hat{g}_k \oplus \mathbb{C}L_{-1}$ of polynomials in t and t^{-1}, is integer graded with the assignments

$$\deg(X \otimes t^n) = n,$$
$$\deg(\zeta) = 0, \quad \zeta \in \mathbb{C},$$
$$\deg(L_{-1}) = -1.$$

Note that $b_{\pm} = g \oplus n_{\pm} \oplus \mathbb{C}$ are naturally Lie subalgebras of $L\hat{g}_k$.

1.2. HIGHEST WEIGHT MODULES

Let V be a (left) g-module. Extend the action of g to b_+ by simply letting n_+ act by zero and $\mathbb{C}$ by multiplication. Then induce:

$$\hat{V} = U(L\hat{g}_k) \otimes_{U(b_+)} V.$$

Here U denotes the universal enveloping algebra. The left $L\hat{g}_k$ module $\hat{V}$ has the following properties:
(i) V is embedded in $\hat{V}$ as the g submodule $1 \otimes V$ and $n_+V = 0$.
(ii) $\hat{V}$ is freely generated over $U(n_-)$ by V.
(iii) For all $v \in \hat{V}$, there exists an m such that $t^m n_+$ acts by zero on v.
(iv) If V is an irreducible highest weight module and $k \notin \mathbb{Q}$ then $\hat{V}$ is irreducible.
The fourth property follows from the Kac–Kazhdan determinant formula, see ref. [1]. By restriction, $\hat{V}$ is a graded $L^{\mathrm{pol}}\hat{g}_k$ module with grading

$$\deg(v) = 0, \qquad v \in V,$$
$$\deg(Nv) = \deg(N), \qquad N \in U(n_-), v \in V.$$

If k is rational, an irreducible highest weight module is obtained as the quotient of $\hat{V}$ by its maximal proper submodule.

1.3. ACTION OF L_{-1}

Let $\{T^a\}$, $a = 1, \ldots, D = \dim(g)$, be an orthonormal basis of g, and set $T^a_n = T^a \otimes t^n$. By property (iii), the action of the formal series

$$L_{-1} = \frac{1}{k + h^\vee} \sum_{n=0}^{\infty} \sum_{a=1}^{D} T^a_{-n-1} T^a_n \tag{1}$$

is well defined on $\hat{V}$, since only finitely many terms contribute. Moreover, the following identity holds in $\mathrm{End}(\hat{V})$:

$$[L_{-1}, X \otimes t^n] = -n\, X \otimes t^{n-1}.$$

If $\hat{V}$ is irreducible, the expression (1) for L_{-1} is unique by Schur's lemma.

Literature for lecture 1. The basic reference for Kac–Moody algebras and their representations is ref. [2].

2. Conformal blocks

2.1. TENSOR PRODUCTS AND MEROMORPHIC FUNCTIONS

Let

$$\mathbb{C}^{[n]} = \mathbb{C}^n - \bigcup_{i<j} \{z \in \mathbb{C}^n | z_i = z_j\}.$$

For $z \in \mathbb{C}^{[n]}$, let $M(z)$ be the algebra of meromorphic functions on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, regular on $\mathbb{P}^1 - \{z_1, \ldots, z_n\}$ and vanishing at infinity. Denote by $g(z)$ the Lie algebra $g \otimes M(z)$ with canonical Lie brackets

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg.$$

We have the injective Lie algebra homomorphism

$$g(z) \to Lg \oplus \cdots \oplus Lg,$$
$$X \otimes f \mapsto \bigoplus_{i=1}^{n} X \otimes f(z_i + t),$$

given by Laurent expansion at the poles.

Now let $V_1, \ldots, V_n$ be g modules. Since $Lg \subset L\hat{g}_k$ (inclusion of vector spaces) and $L\hat{g}_k \oplus \cdots \oplus L\hat{g}_k$ acts on $\bigotimes_{i=1}^{n} \hat{V}_i$, we have a linear map

$$g(z) \otimes \bigotimes_{i=1}^{n} \hat{V}_i \to \bigotimes_{i=1}^{n} \hat{V}_i. \tag{2}$$

But this map defines an action of $g(z)$ on $\bigotimes_{i=1}^{n} \hat{V}_i$. Indeed, in $\mathrm{End}(\bigotimes_{i=1}^{n} \hat{V}_i)$ we have the equation

$$X \otimes f\; Y \otimes g - Y \otimes g\; X \otimes f - [X, Y] \otimes fg = (X, Y)\, k \sum_{i=1}^{n} \mathrm{res}_{t=z_i} f' g \mathrm{d}t,$$

and the right hand side vanishes by the residue theorem.

2.2. CONFORMAL BLOCKS

Let $z \in \mathbb{C}^{[n]}$, and $V_1, \ldots, V_n$ be highest weight g modules. For any left Lie algebra module V denote by V^* the right module $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Define the vector space of conformal blocks as the space of $g(z)$-invariant multilinear functionals

$$\begin{aligned} E(z) &= \mathrm{Hom}_{g(z)}(\otimes \hat{V}_i, \mathbb{C}) \\ &= \{G \in (\otimes_i \hat{V}_i)^* \mid GX = 0 \quad \text{for all } X \in g(z)\}. \end{aligned}$$

The next lemma shows that conformal blocks are completely specified by their value on $\otimes_i V_i$.

Lemma 1. *The restriction map*

$$E(z) \to \mathrm{Hom}_{\mathbb{C}}(\otimes_i V_i, \mathbb{C})$$

is bijective.

The proof is left as an exercise.

2.3. THE FRIEDAN–SHENKER CONNECTION

We now move the points z. The following notation will be used: the action of an element X of a (Lie) algebra on the ith factor of a tensor product of modules will be denoted by $X^{(i)}$. Let $U \subset \mathbb{C}^{[n]}$ be an open set, $M(U)$ the space of meromorphic functions on $U \times \mathbb{P}^1$ whose poles are on the hyperplanes $\{(z,t) | t = z_i\}$ and vanishing for $t = \infty$. Let $g(U) = g \otimes M(U)$. For $X \in g(U)$, denote by $X(z) \in g(z)$ the function $t \mapsto X(z,t)$. A map $G : U \to (\otimes_i \hat{V}_i)^*$ is called holomorphic if $z \mapsto \langle G(z), u \rangle$ is holomorphic on U for all $u \in \otimes_i \hat{V}_i$. Define also the space of holomorphic conformal blocks,

$$\begin{aligned} E(U) = \{ & G : U \to \mathrm{Hom}_{\mathbb{C}}(\otimes_i \hat{V}_i, \mathbb{C}) \text{ holomorphic} | \\ & \langle G(z), X(z)u \rangle = 0 \text{ for all } X \in g(z), u \in \otimes_i \hat{V}_i \}. \end{aligned}$$

If U is not simply connected we allow G in this definition to be many-valued. If $G \in E(U)$ then $G(z) \in E(z)$ for all $z \in U$, and we have an injective restriction map of $E(U)$ to the holomorphic maps with values in $\mathrm{Hom}_{\mathbb{C}}(\otimes_i V_i, \mathbb{C})$.

Consider the trivial infinite rank vector bundle $U \times \mathrm{Hom}_{\mathbb{C}}(\otimes_i \hat{V}_i, \mathbb{C})$ with flat connection

$$\nabla = \sum_{i=1}^{n} \mathrm{d}z_i \nabla_{z_i},$$

$$\nabla_{z_i} G(z) = \partial_{z_i} G(z) - G(z) L_{-1}^{(i)},$$

defined on holomorphic sections $G(z)$, where $\partial_{z_i} G(z)$ is the linear form $u \mapsto \partial_{z_i} \langle G(z), u \rangle$.

Proposition 2. *Let $X \in g(U)$. Then*

$$\nabla_{z_i}[G(z)X(z)] = \nabla_{z_i}G(z)\,X(z) + G(z)\partial_{z_i}X(z).$$

Proof. Let $u \in \otimes_i \hat{V}_i$. We compute

$$\begin{aligned}
&\langle \nabla_{z_i}[G(z)X(z)], u\rangle \\
&= \partial_{z_i}\langle G(z), X(z)u\rangle - \langle G(z), X(z)L_{-1}^{(i)}u\rangle \\
&= \partial_{z_i}\langle G(z), \textstyle\sum_j X(z, z_j + t)^{(j)}u\rangle - \langle G(z), X(z)L_{-1}^{(i)}u\rangle \\
&= \langle \partial_{z_i}G(z), X(z)u\rangle + \langle G(z), \textstyle\sum_j (\partial_{z_i}X)(z, z_j + t)^{(j)}u\rangle \\
&\quad + \langle G(z), [(\mathrm{d}/\mathrm{d}t)X(z, z_i + t)^{(i)} - \textstyle\sum_j X(z, z_j + t)L_{-1}^{(i)}]u\rangle \\
&= \langle \partial_{z_i}G(z), X(z)u\rangle + \langle G(x), \partial_{z_i}X(z)u\rangle - \langle G(z), L_{-1}^{(i)}X(z)u\rangle . \qquad \square
\end{aligned}$$

Corollary 3. *The connection ∇ leaves $E(U)$ invariant, i.e., if $G \in E(U)$ then also $\nabla_{z_i}G \in E(U)$ for all i.*

2.4. THE KNIZHNIK–ZAMOLODCHIKOV EQUATION

The Knizhnik–Zamolodchikov (KZ) equation is the horizontality condition

$$\nabla G = 0.$$

A more explicit formula can be obtained by expressing it in terms of G restricted to $\otimes V_i$. Let $u \in \otimes V_i$, and suppose G is horizontal. Then

$$\begin{aligned}
\partial_{z_i}\langle G(z), u\rangle &= \langle G(z), L_{-1}^{(i)}u\rangle \\
&= \frac{1}{k + h^\vee}\sum_{a=1}^{D}\langle G(z), T_{-1}^{a(i)}T_0^{a(i)}u\rangle .
\end{aligned}$$

Use now the invariance of G under $X = T^a \otimes (t - z_i)^{-1}$. We get the equation for the restriction of $G(z)$ to $\otimes V_i$,

$$\begin{aligned}
\partial_{z_i}G(z) &= \frac{1}{k + h^\vee}\sum_{j\neq i}\frac{\Omega^{ij}}{z_i - z_j}G(z), \\
\langle \Omega^{ij}G, u\rangle &= \langle G, T^{a(i)}T^{a(j)}u\rangle , \qquad u \in \otimes V_i .
\end{aligned}$$

This equation is to be supplemented by the g invariance condition $G(z)X = 0$, for all $X \in g$.

Literature for lecture 2. The content of sections 2.1 and 2.2 is a reformulation of the Ward identities of the Wess–Zumino–Witten model [3], derived in ref. [4]. The formulation of conformal field theory in terms of holomorphic bundles with connection on moduli space was advertised in a broader context in ref. [5].

3. Integral representations, quantum groups

3.1. INTEGRAL REPRESENTATION OF SOLUTIONS FOR sl_2

We turn now to the problem of finding explicit solutions of the KZ equation. We present the result for the technically simplest case of $g = \mathrm{sl}_2$. In terms of the standard generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

we have $\sum_a T^a \otimes T^a = \frac{1}{2} H \otimes H + E \otimes F + F \otimes E$. We use, as before, the notation $X_n = X \otimes t^n$.

If V is a highest weight g module with highest weight Λ, we denote by v_Λ its highest weight vector. Suppose $V_1, \ldots, V_n$ are highest weight modules with highest weights $\Lambda_1, \ldots, \Lambda_n$, respectively. Denote the space of holomorphic conformal blocks on an open set U by $E_{V_1,\ldots,V_n}(U)$.

Let us first consider a special case: set $u_0 = \otimes_i v_{\Lambda_i}$. The KZ equation for $\langle G(z), u_0 \rangle$ reduces to

$$\partial_{z_i} \langle G(z), u_0 \rangle = \frac{1}{k+2} \sum_{j \neq i} \frac{(\Lambda_i, \Lambda_j)}{z_i - z_j} \langle G(z), u_0 \rangle , \tag{3}$$

with normalized solution

$$\langle G_0(z), u_0 \rangle = \prod_{i<j} (z_i - z_j)^{(\Lambda_i, \Lambda_j)/(k+2)} . \tag{4}$$

Extend this solution to $\mathrm{Hom}_{\mathbb{C}}(\otimes V_{\Lambda_i}, \mathbb{C})$ by setting $\langle G_0(z), u \rangle = 0$ for all weight vectors u of weight different from $\sum_i \Lambda_i$. The value of $G_0(z)$ on a general $u \in \otimes_i \hat{V}_i$ is given by a single-valued meromorphic function with possible poles only on the hyperplanes $z_i = z_j$, $i \neq j$, times the right hand side of eq. (4). In particular, the monodromy of $\langle G_0(z), u \rangle$ is independent of u.

We now turn to the case of general weights. The general case can be reduced to the special case just considered, at the cost of introducing new variables. Let α be the simple root of sl_2: $\langle \alpha, H \rangle = 2$. Let r be a non-negative integer. Let W be the highest weight g module with highest weight $-\alpha$. Additionally to the variables $z_1, \ldots, z_n$, we introduce variables $w_1, \ldots, w_r$ and define $\tilde{G}(z, w)$ taking values in $(\otimes_i V_i)^*$, in terms of the solution $G_0 \in E_{V_1,\ldots,V_n,W,\ldots,W}(\mathbb{C}^{[n+r]})$ by the equation

$$\langle \tilde{G}(z, w), u \rangle = \langle G_0(z, w), u \otimes E_{-1} v_{-\alpha} \otimes \cdots \otimes E_{-1} v_{-\alpha} \rangle .$$

Define also $\tilde{G}_j$ by the same formula, but omitting E_{-1} at the jth position ($1 \leq j \leq r$).

Lemma 4. *For any $f \in M(\mathbb{C}^n)$, and $u \in \otimes_i \hat{V}_i$,*

$$\langle \tilde{G}(z,w), (E \otimes f)u \rangle = 0,$$
$$\langle \tilde{G}(z,w), (H \otimes f)u \rangle = 0,$$
$$\langle \tilde{G}(z,w), (F \otimes f)u \rangle = \sum_j \partial_{w_j}[(k+2)f(w_j)\langle \tilde{G}_j(z,w), u \rangle].$$

Proof. Explicit calculation using

$$(F_0 f(w_j) + F_1 f'(w_j))E_{-1}v_{-\alpha} = (k+2)(f(w_j)L_{-1} + f'(w_j))v_{-\alpha}. \quad \square$$

Thus the differential form $\tilde{G}(z,w)\mathrm{d}w_1 \cdots \mathrm{d}w_r$ is $g(\mathbb{C}^{[n]})$ invariant up to exact forms. What this means will be formulated more precisely below. It follows that for every cycle Γ we get a solution of the KZ equation:

$$\langle G_\Gamma(z), u \rangle = \int_\Gamma \langle \tilde{G}(z,w), u \rangle \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_r, \qquad u \in \otimes_i \hat{V}_i. \tag{5}$$

We leave it as an exercise to compute the explicit formula for the integrand, when $u \in \otimes_i V_i$. Take $u = F^{m_1}v_{\Lambda_1} \otimes \cdots \otimes F^{m_n}v_{\Lambda_n}$, with $\sum m_i = r$. The solution is, up to a normalization,

$$\langle \tilde{G}(z,w), u \rangle = \sum_{\sigma \in S(r,m_1,\ldots,m_n)} \prod_{i=1}^{r} \frac{1}{w_i - z_{\sigma(i)}} \Phi(z,w),$$
$$\Phi(z,w) = \prod_{i<j}(z_i - z_j)^{(\Lambda_i,\Lambda_j)/(k+2)} \prod_{i,j}(w_i - z_j)^{-(\alpha,\Lambda_j)/(k+2)} \times \prod_{i<j}(w_i - w_j)^{2/(k+2)}.$$

The sum is over the set $S(r, m_1, \ldots, m_n)$ of maps σ from $\{1, \ldots, r\}$ to $\{1, \ldots, n\}$, such that for all i, the cardinality of $\sigma^{-1}(i)$ is m_i. The expression for G_j is similar, except that $\sum m_i = r - 1$ and $\{1, \ldots, r\}$ is replaced by $\{1, \ldots, r\} - \{i\}$. This solution has weight $\sum_i \Lambda_i - r\alpha$.

3.2. COCHAINS AND CHAINS

Here we assume that $\mathrm{Re}(k) > -2$. Our aim is to find a suitable space of cycles Γ which would give the complete set of solutions of the KZ equation, if we insert it in the integral representation (5).

We consider the situation of the last subsection. We are thus given n highest weight sl_2 modules with highest weights $\Lambda_1, \ldots, \Lambda_n$.

Let $\Omega^*_{n,r}$ be the complex of rational differential forms on $\mathbb{C}^{n+r} \ni (z,w)$, whose poles are on the hyperplanes $\{z_i = z_j\}$, $\{w_i = z_j\}$, vanishing in the z direction

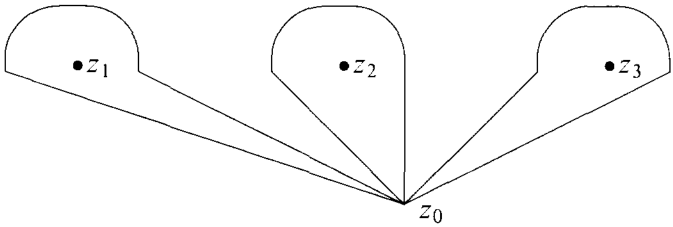

Fig. 1. The loops used to construct cycles.

and with differential

$$d = \sum_{i=1}^{r} dw_i \partial_{w_i}$$

along the w direction.

The symmetric group S_r acts on $\mathbb{C}^{n+r}$ by permutation of the last r variables w. Let $\Omega^*_{n,r\ \mathrm{alt}}$ be the subcomplex of differential forms ω such that $\pi^*\omega = \mathrm{sign}(\pi)\omega$ for all $\pi \in S_r$. Then the space of many-valued differential forms

$$\Delta^*(n,r,\Phi) = \{\Phi\omega \mid \omega \in \Omega^*_{n,r\ \mathrm{alt}}\}$$

[Φ is the function defined in eq. (6)] is in fact a complex. The slightly non-trivial thing to check here is that the differential preserves the regularity as $w_i \to w_j$.

The differential forms we are interested in,

$$\langle \tilde{G}(z,w), u\rangle dw_1 \wedge \cdots \wedge dw_r,$$
$$\textstyle\sum_j (-1)^j \langle \tilde{G}_j(z,w), u\rangle f(w_j) dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_r,$$

are in $\Delta^*(n,r,\Phi)$. Let $\Delta_*(n,r,\Phi) = \mathrm{Hom}(\Delta^*(n,r,\Phi),\mathbb{C})$ be the corresponding homology complex whose differential ∂ is the transposed of d. We want to compute the top homology group

$$H_r(\Delta_*(n,r,\Phi)) = \mathrm{Ker}[\partial : \Delta_r(n,r,\Phi) \to \Delta_{r-1}(n,r,\Phi)]$$

of linear functions on r-forms vanishing on exact r-forms.

Fix a base point $z \in \mathbb{C}^{[n]}$. Let $\gamma_1,\ldots,\gamma_n : [0,1] \to \mathbb{C}$ be closed curves looping around $z_1,\ldots,z_n$ in counterclockwise direction, see fig. 1. Let $r_1,\cdots,r_n$ be non-negative integers. Set

$$\begin{aligned}
\delta_j &= \{s \in [0,1]^j \mid 0 < s_1 < \cdots < s_j < 1\},\\
\delta_{(r_i)} &= \delta_{r_1} \times \cdots \times \delta_{r_n},\\
\gamma_i^j(s) &= (\gamma_i(s_1),\ldots,\gamma_i(s_j)), \qquad s \in \delta_j,\\
\gamma_{(r_i)} &= \gamma_1^{r_1} \times \cdots \times \gamma_n^{r_n} : \delta_{(r_i)} \to \mathbb{C}^{\Sigma r_i}.
\end{aligned}$$

We define a set of chains $\Gamma_{(r_i)}(n,r)$ in $\Delta_r(n,r,\Phi)$ labeled by n-tuples $(r_i) =$

$(r_1, \ldots, r_n)$ with $\sum r_i = r$,

$$\langle \Gamma_{(r_i)}(n,r), \omega \rangle = \int_{\delta_{(r_i)}} \gamma^*_{(r_i)} \omega,$$

with some choice of branch of the many-valued integrand. Let $\overline{\Delta}_r(n,r,\Phi)$ be the subspace of $\Delta_r(n,r,\Phi)$ spanned by the chains $\Gamma_{(r_i)}(n,r)$.

We now define a set of chains $\Gamma_{(r_i)}(n,r)$ in $\Delta_{r-1}(n,r,\Phi)$ labeled by n-tuples $(r_i) = (r_1, \ldots, r_n)$ with $\sum r_i = r - 1$. Let j be the map $(w_1, \ldots, w_{r-1}) \mapsto (w_1, \ldots, w_{r-1}, z_0)$, and define

$$\langle \Gamma_{(r_i)}(n,r), \omega \rangle = \int_{\delta_{(r_i)}} \gamma^*_{(r_i)} j^* \omega,$$

with some choice of branch of the many-valued integrand. In words, what $\Gamma_{(r_i)}(n,r)$ does is: take the piece of the $(n-1)$-form ω not containing $\mathrm{d}w_r$, set $w_r = z_0$, then integrate over the remaining variables w_i. Let $\overline{\Delta}_{r-1}(n,r,\Phi)$ be the subspace of $\Delta_{r-1}(n,r,\Phi)$ spanned by the chains $\Gamma_{(r_i)}(n,r)$. As $\partial(\overline{\Delta}_r) \subset \overline{\Delta}_{r-1}$, we can define a subcomplex $\overline{\Delta}_*(n,r,\Phi)$ by setting all other $\overline{\Delta}_m(n,r,\Phi)$ equal to zero. The following is a somewhat wild form of a conjecture in ref. [6].

Conjecture 5. *The inclusion $\overline{\Delta}_*(n,r,\Phi) \to \Delta_*(n,r,\Phi)$ is a quasi-isomorphism (i.e., it induces an isomorphism in homology).*

What can be shown by a homotopy argument is that the complex $\overline{\Delta}_*(n,r,\Phi)$ computes the Eilenberg–Steenrod singular $H^{\triangle}$-homology [7] of the configuration space of r unordered points on the complement of the union of small open disks about the z_i's, with coefficients in the local system defined by Φ. It is defined as the inverse limit over complements U of compact sets, of relative singular homology groups modulo U.

It is not essential at this point to believe in this conjecture. We can proceed without it, knowing that the the homology of $\overline{\Delta}_*$ in dimension n gives us a space of solutions, which in fact is monodromy invariant.

3.3. THE APPEARANCE OF $U_q(\mathrm{sl}_2)$

Here again, we assume that $\mathrm{Re}(k) > -2$ and set $q = \exp(\pi\mathrm{i}/(k+2))$. Let $U_q(\mathrm{sl}_2)$ be the associative algebra with generators e, f, k^2, k^{-2} and relations $k^2 e = q^2 e k^2$, $k^2 f = q^{-2} f k^2$, $[e,f] = k^2 - k^{-2}$. Let $\Lambda = m\Lambda_0$ be a weight of sl_2 expressed as a multiple of the fundamental weight Λ_0, $m \in \mathbb{C}$.

The quantum Verma module (in the sense of Lusztig [8]) M_Λ contains a highest weight vector v_Λ with

$$e v_\Lambda = 0, \qquad k^2 v_\Lambda = \exp(\pi \mathrm{i} m/(k+2)) v_\Lambda,$$

and has a basis $(v_\Lambda^j)_{j=0}^\infty$:

$$v_\Lambda^j = \frac{f^j}{[j]_q!} v_\Lambda, \qquad [j]_q! = \frac{(q^j - q^{-j}) \cdots (q^2 - q^{-2})(q - q^{-1})}{(q - q^{-1})^j}.$$

If q is a root of unity, there are vanishing denominators in this formula. However, the action of $U_q(\mathrm{sl}_2)$ is well defined, since the matrix elements of generators are regular as q tends to a root of unity. Let us also introduce a grading of $U_q(\mathrm{sl}_2)$ and M_Λ, by setting $\deg(e) = 1, \deg(k^2) = 0, \deg(f) = -1$ and $\deg(v_\Lambda^j) = -j$.

Let us now return to our set of data: n weights $\Lambda_1, \ldots, \Lambda_n$, whose sum is r times the simple root of sl_2. As $U_q(\mathrm{sl}_2)$ is a graded Hopf algebra, we can define graded modules by taking tensor products of graded modules. Denote by $(M)_j$ the space of homogeneous elements of degree j of a module M. We define a complex $\tilde{\Delta}_*(n,r)$, vanishing in all dimensions except in dimension r, $r-1$. Set

$$\tilde{\Delta}_r(n,r) = (M_{\Lambda_1} \otimes \cdots \otimes M_{\Lambda_n})_{-r},$$
$$\tilde{\Delta}_{r-1}(n,r) = (M_{\Lambda_1} \otimes \cdots \otimes M_{\Lambda_n})_{-r+1}.$$

The differential is e.

Theorem 6. *The map* $\overline{\Delta}_*(n,r,\Phi) \to \tilde{\Delta}_*(n,r)$ *defined by*

$$\Gamma_{(r_i)}(n,r) \mapsto c\, v_{\Lambda_1}^{r_1} \otimes \cdots \otimes v_{\Lambda_n}^{r_n}$$

is an isomorphism of complexes, for suitable constants c depending on the choices of branches.

The proof is by explicit computation of the boundary operator.

Thus the quantum group generator e has the interpretation of boundary operator. What is the interpretation of f? It turns out that f can be geometrically identified in terms of "adding a loop". See ref. [6] for details.

The meaning of theorem 6 is that for each singular vector of weight zero in the tensor product of Verma modules, one has a solution of the KZ equation. The monodromy of these solution can be computed as the action of the fundamental group of $\mathbb{C}^{[n]}$ (the pure braid group) on homology.

The result is that one obtains, on the level of the complex, the R-matrix representation of the pure braid group(oid) on the tensor product of quantum Verma modules.

Recent progress [9] shows that the structure described here should have an interesting generalization to Riemann surfaces of arbitrary genus.

Literature for lecture 3. Various derivations of integral representations are known to date. It seems that the most complete result was obtained in ref. [10]. The proof we presented here uses an idea of Cherednik [11]. For quantum groups see ref. [12] and references therein. Early attempts to see a connection between integral representations and quantum groups appeared in refs. [13,14]. We follow ref. [6]. See also refs. [15–17]. The (co)homology of configuration spaces with local coefficient systems in relation with quantum groups has been recently studied in detail in refs. [18–20].

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